

# 3D *hp*-Adaptive Finite Element Simulations of a Magic-T Electromagnetic Waveguide Structure

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## Abstract

This paper employs a 3D *hp* self-adaptive grid-refinement finite element strategy for the solution of a particular electromagnetic waveguide structure known as Magic-T. This structure is utilized as a power divider/combiner in communication systems as well as in other applications. It often incorporates dielectrics, metallic screws, round corners, and so on, which may facilitate its construction or improve its design, but significantly difficult its modeling when employing semi-analytical techniques. The *hp*-adaptive finite element method enables accurate modeling of a Magic-T structure even in the presence of these undesired materials/geometries. Numerical results demonstrate the suitability of the *hp*-adaptive method for modeling a Magic-T rectangular waveguide structure, delivering errors below 0.5% with a limited number of unknowns. Solutions of waveguide problems delivered by the self-adaptive *hp*-FEM are comparable to those obtained with semi-analytical techniques such as the Mode Matching method, for problems where the latest methods can be applied. At the same time, the *hp*-adaptive FEM enables accurate modeling of more complex waveguide structures.

**Keywords:** Finite Element Method (FEM), *hp*-adaptivity, electromagnetic waveguides.

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## 1. Introduction

Electromagnetic (EM) waveguides are of great importance for our daily lives, since they can be found in a variety of devices, including communication systems, radars, satellites, and medical equipment. In this paper, we focus on a specific type of passive rectangular waveguides known as power dividers and power combiners. As suggested by its name, power dividers/combiners are used to either divide or combine the power of a given EM signal and they are employed, among other applications, in RF & Microwave front-ends, and various types of antennas.

One of such power dividers is the so-called Magic-T (also known as magic tee), first described in [1]. Its geometry is depicted in Fig.1. It consists of a “combination” of two orthogonal T-shaped junctions, one in the magnetic H-plane

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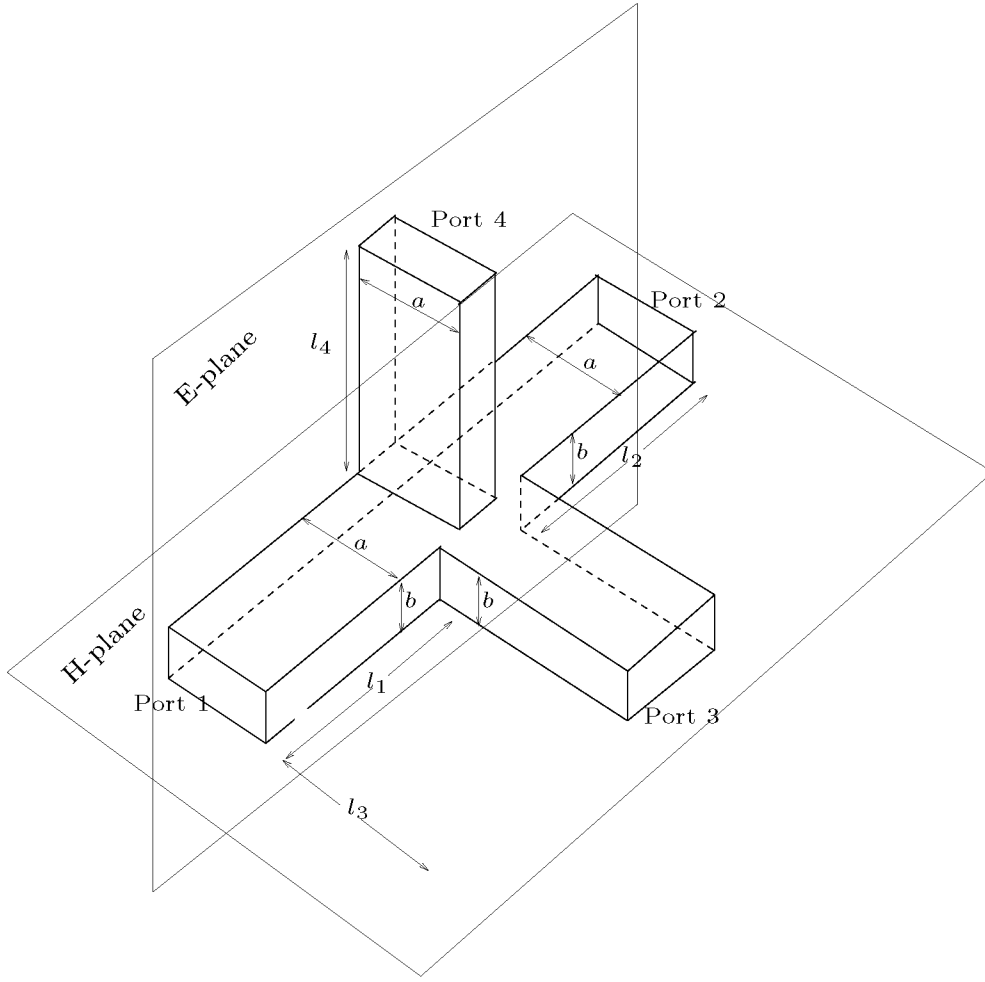


Figure 1: Geometry of a Magic-T waveguide ( $b = a/2$ ,  $l_1 = l_2 = l_3 = l_4 = 1.5a$ )

and one in the electric E-plane. Therefore, when an input signal is injected into the H-port (port 3), the output signal becomes equally distributed between ports 1 and 2 and will be in phase. When an input signal is injected into the E-port (port 4), the output signal also becomes equally distributed between ports 3 and 4, but this time the output signals will have opposite phase (180 degrees out of phase). Its name comes from its *magical* electrical properties. Namely, when ports<sup>1</sup> 3 and 4 are simultaneously matched, that is, when the structure is designed in such a way that the energy reflected at these ports is null, then one can show that ports 1 and 2 become “magically” matched and decoupled.

A Magic-T waveguide, as well as many other waveguides, operates in the monomode region, that is, the excitation frequency is such that when the ports are placed at a certain distance from its center, then only one propagating mode (the so-called  $TE_{10}$  mode) is present with a non-negligible amplitude at the ports. All other modes are *evanescent*, and their amplitude quickly diminishes as the distance from the waveguide discontinuity increases. This implies that the EM field at the ports can be characterized by simply measuring the amount of energy of the propagating mode that is reflected at the excited port and the energy at the other ports. This quantities normalized by the incident energy (“incident power wave”) are related the so-called scattering parameter, also known as S-parameter. More generally, we denote by  $S_{ij}$  to the S-parameter measured at port  $j$  when the  $i$ -th port is excited. Specifically,  $|S_{ii}|^2$  represents

<sup>1</sup>The ports are the apertures at the end of the waveguides.

the ratio between the mean powers of the reflected and incident waves. The argument of  $S_{ii}$  represents the phase shift between the reflected and incident waves. Analogously,  $|S_{ji}|^2$ ,  $j \neq i$  relates the power relation between the transmitted wave at port- $j$  and the incident wave at port- $i$ ; the argument of  $S_{ji}$  gives the phase shift between transmitted and incident waves. The corresponding  $[S]$  matrix with entries  $S_{ij}$  is the so-called scattering matrix.

From the engineering point of view, it is essential to design waveguides with a particular  $[S]$  matrix. In the case of a perfect Magic-T, the desired  $[S]$  matrix is:

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \quad (1)$$

The isolation between ports 3 and 4 comes simply by the properties of the E-plane and H-plane T-junctions. However, as it was mentioned previously, the isolation between ports 1 and 2, requires the *compensation* of the magic-T so ports 3 and 4 are simultaneously matched. This can be accomplished, by the use of internal posts or other techniques without disturbing the symmetry of the structure (c.f., [2]). In any case, due to design limitations (e.g., geometry, size, construction materials and imperfections), it is impossible to achieve the above scattering pattern, and suboptimal designs need to be developed. In this context, the use of numerical simulations is essential both to design improved waveguides under certain restrictions as well as to evaluate the performance of a given waveguide configuration.

Several computational techniques can be used for the analysis of a Magic-T waveguide structure. First, we have analytic (or semi-analytic) techniques such as the Mode Matching (MM) method (c.f., [3, 4]), which are computationally very efficient, and they can be employed to design several devices with given specifications using optimization procedures [5, 6]. These methods have severe limitations in terms of the geometries they can handle. Thus, hybrid extensions as those of [7–10]) have been developed in the last years. They overcome many of the limitations of pure analytic (or semi-analytic) techniques but still need “separable” geometries, which limits the type of geometries they can handle. Specifically, they cannot support the presence of arbitrary objects such as screws or curved geometries that can appear as a result of the waveguide construction and manufacturing. Thus, the use of pure numerical methods that can analyze the structure as a whole, taking into account the effect of round corners due to finite radii of milling tools, the influence/sensitivity to position of tuning elements (posts, screws, etc) not oriented in the coordinate axis, losses, and so on, is preferred.

The main advantage of pure discretization methods such as finite element methods (FEM) and Integral Equation (IE) approaches is their flexibility to model arbitrary materials and geometries. Moreover, even for devices designed with semi-analytic methods, they need to be finally validated with a pure numerical technique that can analyze the structure as a whole, taking into account the effect of round corners due to finite radii of milling tools, the influence/sensitivity to position of tuning elements (posts, screws, etc), losses, and so on. In this context, FEM has demonstrated to be a powerful and versatile tool. However, high accuracy is not always easy to achieve for conventional FEM when dealing with field singularities, high-contrast material data, etc. Thus, semi-analytic results (when available) are typically considered as a reference for engineering analysis.

In this work, the use of a highly-accurate three-dimensional (3D) fully automatic *hp*-adaptive FEM for the characterization of a Magic-T waveguide is proposed. This method combines the geometrical flexibility of a FEM with an accuracy that is often superior to that provided by semi-analytical methods, as it will be shown in Section 4. Such a high-accuracy delivered by the *hp*-FEM is due to the exponential convergence of the error on optimally designed *hp*-meshes. These meshes adapt to the problem of interest, and are delivered automatically (without any user-interaction) by the *hp*-adaptive strategy. This grid-refinement strategy efficiently deals with different types of singularities. Moreover, the dispersion (pollution) error is also automatically minimized by increasing the polynomial order of approximation (see [11, 12]), and therefore, the method is also capable of solving efficiently high-frequency waveguide problems.

This paper constitutes an extension of our previous 2D paper [13] to the case of 3D waveguides. The level of complexity in 3D (in both, mathematical and computational senses) of the *hp* FE method, and specifically of the *hp*-adaptivity, is several orders of magnitude higher than in 2D. This is due, among other reasons, to the higher number of “choices” available to decide the next *hp*-mesh at each step of the adaptivity. Also, it is due to the (higher)

computational order of complexity with polynomial order  $p$  involved in the computations. In addition, a much larger number of unknowns are usually involved in 3D problems when compared with 2D problems.

The particular  $hp$ -adaptive algorithm used in this paper builds upon previous versions developed by L. Demkowicz and his collaborators [14–18]. In the version used in this work, we fixed a number of errors and implementation problems, we increased its performance for realistic 3D computations, and we implemented proper post-processing schemes to compute the S-matrix.

The paper is organized as follows. Section 2 describes the mathematical formulation of the problem. Then, the  $hp$ -method including the self-adaptive refinement strategy is detailed in Section 3. Section 4 outlines the main results for a Magic-T waveguide structure obtained with the  $hp$ -adaptive method. Finally, some conclusions are depicted in Section 5.

## 2. Mathematical Formulation

The EM phenomena are governed by a set of four first order partial differential equations (PDEs) known as Maxwell's equations. For a time-harmonic nonzero frequency excitation ( $\omega \neq 0$ ), two of those equations (the so-called Gauss's laws) become redundant, and the remaining two equations (Ampere's and Faraday's laws) along with the corresponding boundary conditions provide a unique solution for the EM field:

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} + \mathbf{J}^{imp} & (\text{Faraday's law}) \\ \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} & (\text{Ampere's law})\end{aligned}$$

In the above equations,  $\mathbf{E}$  and  $\mathbf{H}$  stand for the electric and magnetic field, respectively,  $j = \sqrt{-1}$  is the imaginary unit,  $\epsilon$  and  $\mu$  are the electric permittivity and permeability of the medium, respectively, and  $\mathbf{J}^{imp}$  is the impressed volumetric electric current, which is zero in our case.

The above system of two first order PDE's can be further simplified to a single second order PDE in terms of magnetic field  $\mathbf{H}$  by taking the curl of Ampere's law pre-multiplied by  $1/\epsilon$  and using Faraday's law to obtain the so called reduced wave equation:

$$\nabla \times \frac{1}{\epsilon} \nabla \times \mathbf{H} - \omega^2 \mu \mathbf{H} = 0 \quad (2)$$

Assuming  $\text{TE}_{10}$  mode excitation and excitation frequency in the monomode region, the following boundary condition can be used at the ports, namely:

$$\hat{\mathbf{n}} \times \frac{1}{\epsilon} \nabla \times \mathbf{H} + j \frac{\omega^2 \mu}{\beta_{10}} \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H} = \mathbf{U}^{\text{in}}, \text{ at the port boundaries } (\Gamma_p) \quad (3)$$

where  $\hat{\mathbf{n}}$  is the unit outward vector normal to  $\Gamma$ ,  $\beta_{10}$  is the phase constant of the  $\text{TE}_{10}$ , and  $\mathbf{U}^{\text{in}}$  is defined as

$$\mathbf{U}^{\text{in}} = 2j \frac{\omega^2 \mu}{\beta_{10}} \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}^{\text{in}} \quad (4)$$

with  $\mathbf{H}^{\text{in}}$  being the incident magnetic field at the port. When  $\mathbf{H}^{\text{in}} = \mathbf{0}$ , the above boundary conditions at the ports (3) become absorbing boundary conditions for the *monomode propagation* case. The remaining modes are *evanescent* when in the monomode region, and they quickly diminish in amplitude as they travel towards the ports. Thus, it is necessary to truncate the FEM domain at a certain distance from the waveguide discontinuity so the amplitude of the *evanescent* modes becomes negligible. The waveguide walls are modelled with perfect electric boundary conditions:

$$\hat{\mathbf{n}} \times \frac{1}{\epsilon} \nabla \times \mathbf{H} = 0, \quad \text{at perfect electric conductors } (\Gamma_N) \quad (5)$$

The FEM variational formulation is obtained by multiplying (2) by a test function  $\mathbf{F} \in \mathbf{H}(\text{curl}, \Omega)$ , integrating by parts, and incorporating the boundary conditions to obtain:

$$\int (\nabla \times \mathbf{F}) \cdot \left( \frac{1}{\epsilon} \nabla \times \mathbf{H} \right) d\Omega - \omega^2 \int \mathbf{F} \cdot \mu \mathbf{H} d\Omega + j \frac{\omega^2 \mu}{\beta_{10}} \int_{\Sigma_{\Gamma_p}^i} (\hat{\mathbf{n}} \times \mathbf{F}) \cdot (\hat{\mathbf{n}} \times \mathbf{H}) d\Gamma = 2j \frac{\omega^2 \mu}{\beta_{10}} \int_{\Gamma_p^{\text{in}}} (\hat{\mathbf{n}} \times \mathbf{F}) \cdot (\hat{\mathbf{n}} \times \mathbf{H}^{\text{in}}) d\Gamma. \quad (6)$$

We note that the above formulation is given in terms of the magnetic field, although a dual formulation in terms of the electric field can also be derived in a similar way.

### 3. *hp* Finite Elements and Automatic Adaptivity

With each finite element, we associate element size  $h$  and order of approximation  $p$ . In the  $h$ -adaptive version of FEM, element size  $h$  may vary from element to element, while order of approximation  $p$  is fixed (usually  $p=1,2$ ). In the  $p$ -adaptive version of the FEM,  $p$  may vary locally, while  $h$  remains constant throughout the adaptive procedure. Finally, a true  $hp$ -adaptive version of the FEM allows for local variations of both  $h$  and  $p$ .

The  $hp$ -FEM constitutes a flexible method suitable for solutions containing both smooth and non-smooth components in different areas of the computational domain [18, 19]. Solutions incorporating smooth components and a variety of singularities are typical in waveguide applications, and thus, the  $hp$ -FEM is especially well-suited for these applications. While the spectral  $p$ -FEM provides accurate approximations of the smooth components of the solution and minimizes the dispersion error in the case of wave propagation problems (see [11, 12, 20]), the  $h$ -FEM is intended to accurately approximate singular solutions, as it occurs in the case of re-entrant corners in EM or in areas of the computational domain where three or more materials meet at a point.

To determine an optimal distribution of element size  $h$  and polynomial order of approximation  $p$ , we shall employ a self-adaptive refinement strategy based on the iterative scheme described in [16, 18]. At each step, given an arbitrary  $hp$ -grid, we first perform a global and uniform  $hp$ -refinement to obtain the  $h/2, p+1$ -grid. Second, we approximate the error function in the  $hp$ -grid by evaluating the difference between the solutions associated to the  $hp$ - and  $h/2, p+1$ -grids. If the error exceeds a user-prescribed tolerance error, then we project the error function to guide optimal refinements over the  $hp$ -grid, and we iterate the process. Once the prescribed tolerance error has been met, we deliver the  $h/2, p+1$ -grid as the ultimate solution of the problem. This three-dimensional refinement strategy has been proved to be efficient, robust, and highly accurate [18] in many applications.

For optimal selection between  $h$  and  $p$  refinements, we utilize the projection-based interpolation (see [21, 22]) operators to project the solution associated to the  $h/2, p+1$ -grid into a sequence of coarser grids containing more unknowns than those of the  $hp$ -grid, and we compare results of those projections to determine the one providing the largest error decrease rate per added unknown. To evaluate the error of each projected solution, we employ the energy norm. To make this optimization problem tractable from the computational point of view, we first determine optimal refinements for edges, then for faces, and finally for element interiors. Critical to the success of this projection based approach is the commutativity of the so-called *de Rham* diagram, which is used to ensure stability and convergence of the final formulation, avoiding the appearance of spurious (unphysical) modes. The simulator utilizes hexahedral edge (Nedelec) elements of variable order of approximation to discretize  $\mathbf{H}(\text{curl})$ .

Our  $hp$ -adaptive FEM supports local anisotropic refinements in both  $h$  and  $p$ . Support of anisotropic refinements dramatically increases the implementation complexity, but it is essential to achieve optimal convergence rates in the case of edge singularities or in the presence of boundary layers. Both situations occur in our Magic-T problem of interest. To ensure continuity, we enforce the 1-irregularity rule and use the constrained approximation [23].

As mentioned above, the main advantage of the  $hp$ -FEM resides on the proof (see [24–30]) showing that it can achieve exponential convergence for elliptic problems with a piecewise analytic solution, whereas  $h$ - or  $p$ -FEM converge at best algebraically. To attain that goal, we have further extended the existing  $hp$ -adaptive strategy for the case of waveguide problems.

For the analysis of the Magic-T, we have implemented impedance boundary conditions as well as several post-processing routines for computing the S-parameters, graphics, and so on. We have also fixed a few problems and accelerated a number of key features within the software such as the computation of the projections during the  $hp$ -adaptivity.

### 4. Numerical Results

To illustrate the results delivered by the fully automatic  $hp$ -adaptivity, we display in Fig. 2 two meshes corresponding to the 3rd and 13th iteration of the adaptive procedure for the Magic-T structure excited by port 1. As expected, we observe heavy  $p$ -refinements in those areas of the domain where solution is smooth (away from the junctions)

Table 1: Magnitude of scattering parameters for the Magic-T.

	$ S_{11} $	$ S_{21} $	$ S_{31} $	$ S_{41} $	$ S_{33} $	$ S_{43} $	$ S_{44} $
Iter. 1	0.0975	0.4256	0.5687	0.3525	0.4202	$10^{-15}$	0.4731
Iter. 3	0.1099	0.5727	0.5883	0.6270	0.5535	$3 \times 10^{-15}$	0.4604
Iter. 10	0.0536	0.5086	0.5858	0.6291	0.5601	$10^{-13}$	0.4565
Iter. 12	0.0524	0.5083	0.5858	0.6291	0.5601	$3 \times 10^{-15}$	0.4566
MM	0.0521	0.5083	0.5859	0.6290	0.5599	$10^{-15}$	0.4568

and additional  $h$ -refinements where a singular behavior of some components of the solution is expected (close to the junctions).

This adaptive strategy results in exponential convergence rates, as illustrated in Fig. 3, where we display the convergence history (up to an error around 0.3%) for the Magic-T structure excited by ports 3 and 4.

The tenth mesh delivered by the  $hp$ -adaptivity when exciting port 3 is shown in Fig. 4 (left panel). The ninth mesh delivered when exciting port 4 is displayed in Fig. 4 (right panel). Both meshes correspond to an energy error around 1%. The initial mesh was selected to be as coarse as possible in terms of  $h$  and with uniform  $p = 2$ . The large difference of the meshes at this error level is due to the following. When exciting port 3, the T-junction of the H-plane is excited (ports 1,2,3). And when port 4 is excited, the T-junction of the E-plane is excited (ports 1,2,4). The field of the H-plane T-junction is quite different to the one of the E-plane T-junction and so are the meshes delivered by the adaptivity.

The isolation between ports 3 and 4 is illustrated in Fig. 5 in which the magnitude of the magnetic field in the structure when exciting port 3 is displayed. Specifically, the component of the displayed magnetic field corresponds to the longitudinal component of port 3 and the transverse component of ports 1 and 2. No field is observed in port 4. Note the stationary wave pattern in waveguide of port 3 due to existence of a non-negligible reflected wave in addition to the incident wave. No stationary wave is observed in the output waveguides of ports 1 and 2 as there is only one wave propagating outward on each of the transmitted ports.

As the structure has four ports, there is a maximum of sixteen  $S$ -parameters to be computed. However, by reciprocity  $S_{ji} = S_{ij}$ . In addition, the symmetry of the Magic-T forces other relations, e.g.,  $S_{41} = S_{42}$ . In summary, the  $[S]$  matrix can be defined in terms of only seven parameters as follows:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{21} & S_{31} & S_{41} \\ S_{21} & S_{11} & S_{31} & S_{41} \\ S_{31} & S_{31} & S_{33} & S_{43} \\ S_{41} & S_{41} & S_{43} & S_{44} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad (7)$$

The values computed by  $hp$ -FEM of some of the  $S$ -parameters obtained when exciting ports 3 and 4 are shown in Tab. 1. Those values are compared with the corresponding ones obtained with a MM technique. Only the magnitudes are shown; the  $S_{ij}$  phase difference between the results delivered by the  $hp$ -FEM and MM is around  $0.01^\circ$ . Only four significant digits are shown in the table as the MM results are presumed to have no more than four digits of accuracy. Note that  $S_{43} = 0$ . This is the result obtained numerically (close to machine precision) with MM. Similar values are provided by the  $hp$ -FEM. Note that scattering matrix does not correspond to a scattering matrix of a perfectly compensated Magic-T, since our waveguide structure has intentionally been left uncompensated.

## 5. Conclusions

A  $hp$ -adaptive FE strategy has been employed for the simulation of a Magic-T structure. Computation of the scattering matrix used to characterize the electromagnetic behavior of the Magic-T structure has been implemented as a post-processing of the solution. Grids delivered by the  $hp$ -adaptive strategy indicate an optimal (or quasi-optimal)

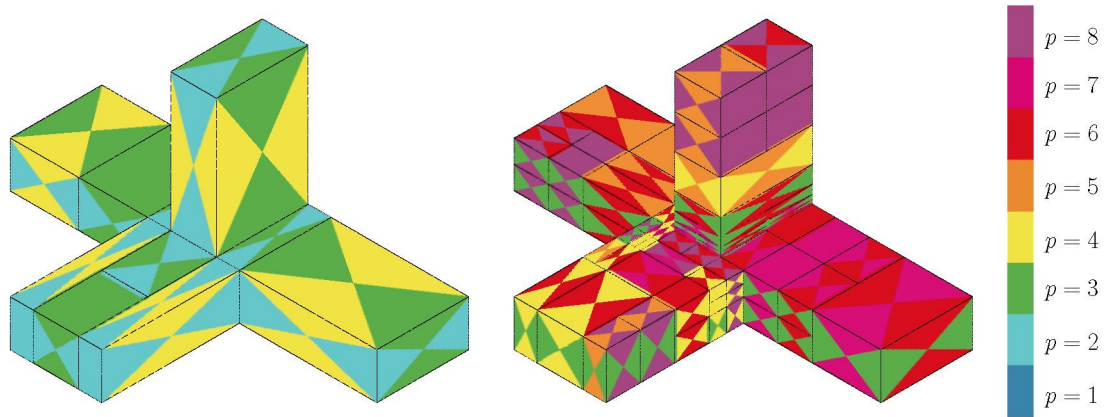


Figure 2: Two meshes produced by the self-adaptive  $hp$ -refinement strategy for the Magic-T problem when excited by port 1. Left panel: Mesh after 3 iterations. Right panel: Mesh after 13 interactions. Different colors indicate different polynomials orders of approximation.

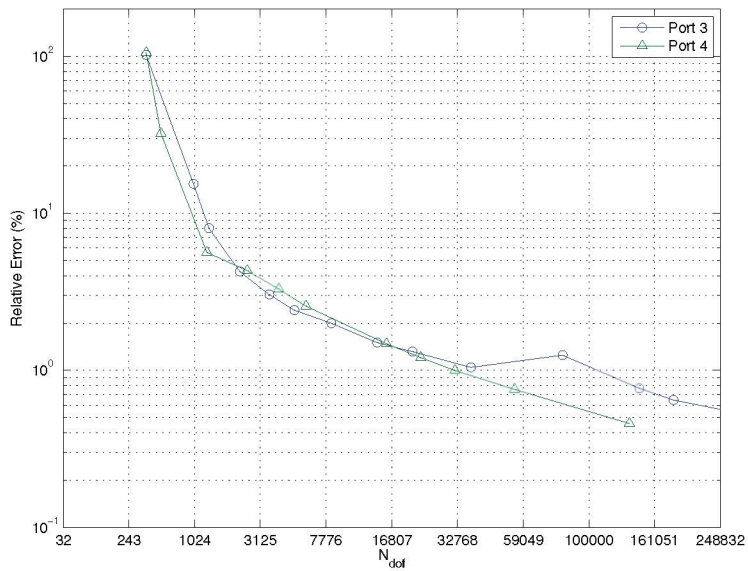


Figure 3: Convergence history for Magic-T.

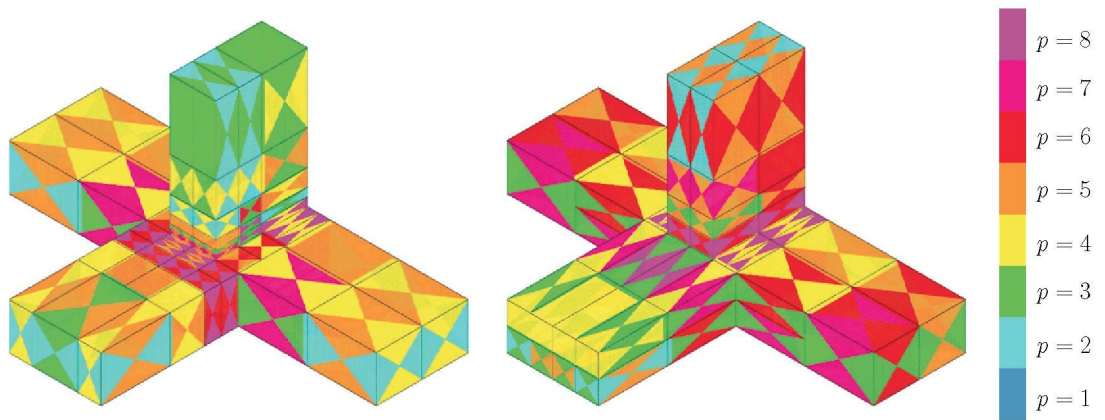


Figure 4: Left panel: Mesh of 10th iteration for Magic-T (excitation by port 3). Right panel: Mesh of 9th iteration for Magic-T (excitation by port 4).

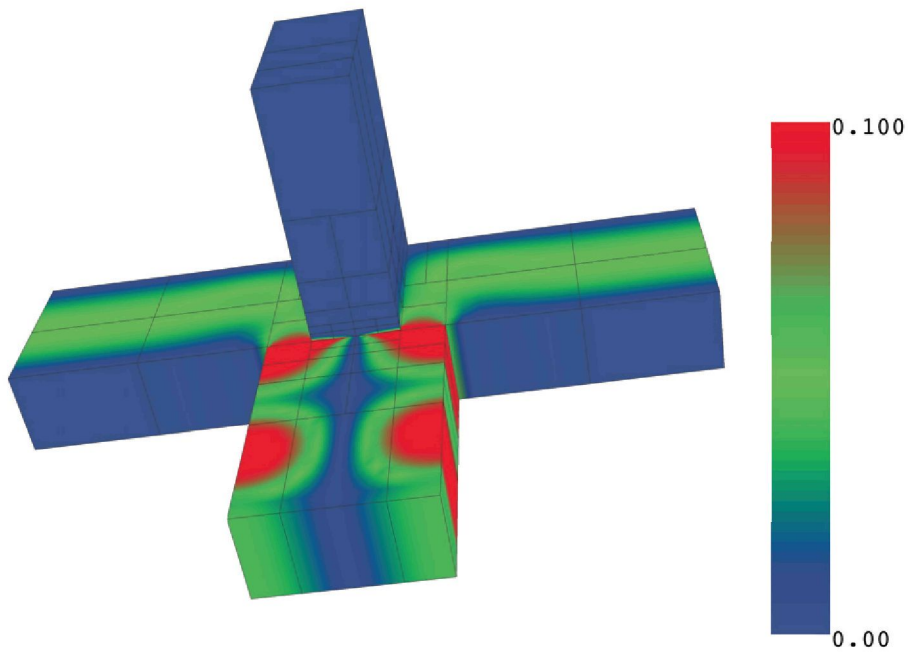


Figure 5: Magnitude of x-component of magnetic field for Magic-T (excitation by port 3).



distribution of element sizes  $h$  and polynomial orders of approximation  $p$  throughout the computational grid. These results have been confirmed by the excellent convergence curves, and the final simulations which have a comparable accuracy to those obtained with semi-analytical methods. At the same time, the  $hp$ -FEM maintains the flexibility of any FEM, which can easily account for different geometries, materials, and construction or design artifacts.

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